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# The exchange algebra for Zamolodchikov and Fateev's parafermionic theories

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**Abstract.** The concept of an exchange algebra has recently been introduced by Rehren and Schroer in the context of two-dimensional conformal field theories to give an algebraic setting to both the dynamics and the locality requirement. Labelling the conformal families with two indices and assuming an interpolating scheme for one of the fields, it is shown that the braiding matrices for a subset of fields in Zamolodchikov's and Fateev's (ZF) parafermionic theories containing all the order parameters are identical to those of the diagonal minimal models. We recover the full spectrum of these theories modulo integers from the phase condition of the exchange algebra even though the subset does not include the parafermionic currents.

## 1. Introduction

Several methods have been proposed in the past two years to tackle one of the remaining problems in conformal quantum field theories: the problem of classification. Most of them are similar and have an important algebraic content (polynomial equations [1], quantum group approach [2], exchange algebra [3–5], etc). Even though they are aimed at the classification problem, they can also be used to obtain quantitative information about physical field theories.

The concept of exchange algebra has been (re-)introduced in the context of two-dimensional conformal field theories to characterize the algebraic structure of the light-cone interpolating fields (also known as chiral vertex operators in Euclidean space). More precisely it states how these fields can be braided ('commuted'). The braiding matrices satisfy simple equations, one of which is a Yang–Baxter-type equations. In a simple calculation Rehren [4] showed how these structural equations contain much information. Indeed, using one further hypothesis, he was able to calculate the braiding properties of some basic fields and to obtain the spectrum of both the minimal models and the WZW theories. His hypothesis was the existence of a field $\alpha$ whose fusion rules, for some labelling of the conformal families, are $[\alpha][l] = [l-1] \oplus [l+1]$ for $l \in \mathbb{Z}_{m-1}$. (This was referred to, in [4], as the interpolation scheme of these models.) Let us stress that nothing was used from the representation theory of the Virasoro

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or affine algebras, although the proofs of the basic properties of the exchange algebra rest upon the $\widetilde{\mathrm{SL}(2,\mathbb{R})}_+ \times \widetilde{\mathrm{SL}(2,\mathbb{R})}_-$ invariance of the theories.

In the present paper we apply these ideas to another set of quantum field theories (namely the Zamolodchikov and Fateev (ZF) parafermionic theories [6]). The defining equations of the exchange algebra will again show their power by giving the full parafermionic spectrum, among other things. Section 2 surveys definition of the exchange algebra by Rehren and Schroer. Section 3 applies these ideas to a specific scheme of interpolation; these steps are the following: description of the set of fields and their interpolationg scheme (section 3.1), connection between the braiding matrices for this set and those for the diagonal minimal models (section 3.2) and solution of the phase condition (section 3.3). In section 4, we explain the relationship between the set chosen in section 3.1 and the field content of parafermionic theories and draw some conclusions.

## 2. The exchange algebra

In this section we write down the defining equations for the exchange algebra [3].

Conformal field theories can be constructed on Hilbert spaces which are direct sums of irreducible representations of a symmetry algebra $\mathcal{A} \times \bar{\mathcal{A}}$. Both subalgebras $\mathcal{A} \otimes \mathbf{1}$ and $\mathbf{1} \otimes \bar{\mathcal{A}}$ are associated with one of the light-cones and contain the Virasoro algebra. As usual, these representations are taken to be unitary and the energy–momentum tensor is supposed to be conserved and of conformal dimension 2. We add the further requirement that the Hilbert space contains only a finite number of irreducible representations (irreps) of $\mathcal{A}$ and $\bar{\mathcal{A}}$. Hence

$$\mathcal{H} = \bigoplus_{\alpha,\bar{\alpha}} \mathcal{H}_\alpha \otimes \mathcal{H}_{\bar{\alpha}}$$

where $\mathcal{H}_\alpha$ and $\mathcal{H}_{\bar{\alpha}}$ are irreps of $\mathcal{A}$ and $\bar{\mathcal{A}}$ respectively and where the pair $(\alpha, \bar{\alpha})$ takes its values in a finite set. (In the minimal models, $\mathcal{A}$ and $\bar{\mathcal{A}}$ are the Virasoro algebra and in the WZW models, a Kac–Moody algebra.) In general, the chiral algebra $\mathcal{A}$ could be large enough for a given irrep to contain several irreducible highest weight representations of the Virasoro algebra whose weights are *not* necessarily equal to modulo $\mathbb{Z}$. We shall come back to this problem in the last section. For the rest of the present one, we suppose that all the irreps of the Virasoro algebra in a given irrep of $\mathcal{A}$ have the same highest weight mod $\mathbb{Z}$.

Each primary physical field $\Phi(x^+, x^-)$ is characterized by a pair of conformal dimensions $(h_\Phi, \bar{h}_\Phi)$ and can be written as a sum of products of interpolating fields:

$$\Phi(x^+, x^-) = \sum_{(\alpha,\beta),(\bar{\alpha},\bar{\beta})} g^{(\alpha,\beta),(\bar{\alpha},\bar{\beta})}_{(h_\Phi,\bar{h}_\Phi)} \; {}_\alpha \overset{h_\Phi}{\perp}_\beta \, (x^+) \; {}_{\bar{\alpha}} \overset{\bar{h}_\Phi}{\perp}_{\bar{\beta}} \, (x^-)$$

where ${}_\alpha \overset{h_\Phi}{\perp}_\beta \, (x^+)$ is an operator which is zero on all $\mathcal{H}_\gamma$ except $\mathcal{H}_\beta$ and whose image is in $\mathcal{H}_\alpha$. In other words, if $P_\gamma$ denotes the projector on $\mathcal{H}_\gamma$, then ${}_\alpha \overset{h_\Phi}{\perp}_\beta = P_\alpha \, {}_\alpha \overset{h_\Phi}{\perp}_\beta \, P_\beta$. The interpolating fields ${}_\alpha \overset{h_\Phi}{\perp}_\beta$ are often called *chiral vertex operators* (in Euclidean space). The $g^{(\alpha,\beta),\,(\bar\alpha,\bar\beta)}_{(h_\Phi,\bar h_\Phi)}$ are complex constants that depend on the normalization of the interpolating fields ${}_\alpha\overset{h_\Phi}{\perp}{}_\beta$.

Any $n$-point correlation functions $\langle\Phi_1(x_1^+,x_1^-)\Phi_2(x_2^+,x_2^-)\ldots\Phi_n(x_n^+,x_n^-)\rangle$ can be decomposed in terms of conformal blocks[7]:

$$\langle\Phi_1(x_1^\pm)\ldots\Phi_n(x_n^\pm)\rangle = \sum_{\{\beta\},\{\bar\beta\}} F_{\{\beta\}}\,(x_1^+,\ldots,x_n^+)\,\bar F_{\{\bar\beta\}}\,(x_1^-,\ldots,x_n^-)$$

where the $\{\beta\}$s are multi-indices labelling the various subspaces $\mathcal{H}_{\beta_i}$ the fields are interpolating from and to

$$F_{\{\beta\}}(x_1^+,\ldots,x_n^+) = \langle{}_0\overset{\phi_1}{\perp}{}_{\beta_1}\overset{\phi_2}{\perp}{}_{\beta_2}\cdots{}_{\beta_{n-1}}\overset{\phi_n}{\perp}{}_0\rangle.$$

To insure locality of the theory, the conformal blocks $F$ and $\bar F$ have to satisfy intricate conditions. These clearly cannot be formulated in terms of a single light-cone coordinate. To help write down the locality requirements, Rehren and Schroer introduced an exchange matrix which characterizes the exchange of two interpolating fields in the conformal blocks. The definition is:

$${}_{\beta_0}\overset{\alpha_1}{\perp}{}_{\beta_1}\,(x_1^+)\,{}_{\beta_1}\overset{\alpha_2}{\perp}{}_{\beta_2}\,(x_2^+) = \sum_{\beta_1'}\left[R^{(\beta_0\,\beta_2)+}_{(\alpha_0\,\alpha_2)}\,(x_1^+,x_2^+)\right]_{\beta_1,\beta_1'}\,{}_{\beta_0}\overset{\alpha_2}{\perp}{}_{\beta_1'}\,(x_2^+)\,{}_{\beta_1'}\overset{\alpha_1}{\perp}{}_{\beta_2}\,(x_1^+). \tag{2.1}$$

(From now on, the '$\pm$' indices will be dropped. The requirement of locality ties the matrices $R^+$ and $R^-$ and the $g^{(\alpha,\beta),\,(\bar\alpha,\bar\beta)}_{(h_\Phi,\bar h_\Phi)}$. However, we shall not discuss this relationship here.) The following three basic properties are satisfied by $R$:

(i) $\left[R^{(\beta_0\,\beta_2)}_{(\alpha_1\,\alpha_2)}\,(x_1,x_2)\right]_{\beta_1,\beta_1'}$ depends on $x_1$ and $x_2$ only through their relative positions. Moreover, if $x_{12}=\mathrm{sgn}(x_1-x_2)=\pm$, then

$$\left[R^{(\beta_0\,\beta_2)}_{(\alpha_1\,\alpha_2)}\,(x_{12})\right]^{-1} = R^{(\beta_0\,\beta_2)}_{(\alpha_1\,\alpha_2)}\,(x_{21}). \tag{2.2}$$

Hereafter

$$\left[R^{(\beta_0\,\beta_2)}_{(\alpha_1\,\alpha_2)}\right] \equiv \left[R^{(\beta_0\,\beta_2)+}_{(\alpha_1\,\alpha_2)}\right].$$

(ii) *Phase condition.* $\left[R^{(\beta_0\,\beta_2)}_{(\alpha_1\,\alpha_2)}\right]$ and $\left[R^{(\beta_0\,\beta_2)}_{(\alpha_2\,\alpha_1)}\right]$ are related through the following relationship:

$$\sum_{\beta_1'}\left[R^{(\beta_0\,\beta_2)}_{(\alpha_1\,\alpha_2)}\right]_{\beta_1,\beta_1'}\,\left[R^{(\beta_0\,\beta_2)}_{(\alpha_2\,\alpha_1)}\right]_{\beta_1',\beta_1''}\,\exp\left(2\pi\mathrm{i}\left(h_{\beta_1}+h_{\beta_1'}-h_{\beta_0}-h_{\beta_2}\right)\right) = \delta_{\beta_1\,\beta_1''}. \tag{2.3}$$

(iii) *Braid relation.* The exchange matrices satisfy

$$\sum_{\beta_1''}\left[R^{(\beta_0\,\beta_2)}_{(\alpha_1\,\alpha_2)}\right]_{\beta_1,\beta_1''}\,\left[R^{(\beta_1''\,\beta_3)}_{(\alpha_1\,\alpha_3)}\right]_{\beta_2,\beta_2'}\,\left[R^{(\beta_0\,\beta_2')}_{(\alpha_2\,\alpha_3)}\right]_{\beta_1''\,\beta_1'}$$
$$= \sum_{\beta_2''}\left[R^{(\beta_1\,\beta_3)}_{(\alpha_2\,\alpha_3)}\right]_{\beta_2\beta_2''}\,\left[R^{(\beta_0\,\beta_2'')}_{(\alpha_1\,\alpha_3)}\right]_{\beta_1\beta_1'}\,\left[R^{(\beta_1'\,\beta_3)}_{(\alpha_1\,\alpha_2)}\right]_{\beta_2''\beta_2'}. \tag{2.4}$$

The relationships (2.1)–(2.4) define the exchange algebra.

For the minimal and WZW models, Rehren and Schroer [3]–[4] gave the solution of these equations for one fundamental field (see section 3.2). Recently, Felder *et al* [8] have extended their solution to the whole field content of the minimal models.

## 3. The exchange algebra for a new interpolating scheme

In this section, we compute the exchange algebra and the associated spectrum for another interpolating scheme. In the next section, the set of fields and fusion rules considered here will be shown to be intimately related to parafermionic theories; namely, they will be identified with a subset of the fields in these theories containing the order parameters. We delay the physical discussion to this latter section.

### *3.1. The interpolating scheme*

Consider the following set of conformal families

$$A = \{[\phi^l_m],\ 0 \leqslant l \leqslant N,\ m \geqslant l \text{ and } m = l \bmod 2\} \tag{3.1}$$

together with the following fusion rules

$$[\phi^{l_1}_{m_1}][\phi^{l_2}_{m_2}] = \sum_{\substack{l=|l_1-l_2| \\ l=l_1+l_2 \bmod 2}}^{\min\{l_1+l_2, 2N-l_1-l_2\}} [\phi^l_{m_1+m_2}] \tag{3.2}$$

and generating by action on the vacuum a Hilbert space of the form

$$\mathcal{H} = \bigoplus_{l,m} \mathcal{H}_{(l,m)}. \tag{3.3}$$

As mentioned previously, it will be argued in section 4 that $A$ corresponds to a subset of mutually local fields in ZF parafermionic theories, namely that it is the smallest ensemble containing the conformal families of the order parameters ($[\phi^k_k]$, $k = 0, \ldots, N-1$) and closed under the OPE. Any family of $A$ is generated by OPE of fields belonging to the family $[\phi^1_1]$:

$$[\phi^1_1][\phi^l_m] = [\phi^{l-1}_{m+1}] + [\phi^{l+1}_{m+1}] \tag{3.4}$$

with the obvious omission when $l-1 = -1$ and $l+1 = N+1$. (The fusion rule of the field $[\phi^1_1]$ will be referred to as the 'interpolating scheme' in what follows. See [4].) Hence the product $[\phi^1_1][\phi^1_1]$ contains the families $[\phi^0_2]$ and $[\phi^2_2]$; from the latter one gets $[\phi^1_3]$ and $[\phi^3_3]$ and so on. The exchange algebra of any of the fields in $A$ can then be obtained from those of the field $\phi^1_1$ which plays a similar role to the field $\phi_{(1,2)}$ in the minimal models and considered by Rehren [4]. For this reason the exchange matrices for the interpolating components of $\phi^1_1$ are called the *fundamental braiding matrices.* Due to the fusion rules (3.4), there are only two sets of interpolating fields building $\phi^1_1$ on each light-cone:

$$_{(l-1,m+1)} \overset{(1,1)}{\perp} {}_{(l,m)} \qquad \text{and} \qquad _{(l+1,m+1)} \overset{(1,1)}{\perp} {}_{(l,m)}. \tag{3.5}$$

Hence, the scheme (3.4) is similar to the one used in [4]; the difference lies in the additive charge $m$ that specifies the transformation properties of the field under the action of $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ (discussed later). We shall show that the fundamental exchange matrices are independent of $m$ and hence are identical to those of the minimal models. The charge structure, however, introduces a new freedom to the spectrum of conformal weights compatible with the phase condition of the exchange algebra. This freedom allows for the full spectrum of the parafermionic theories to be reproduced.

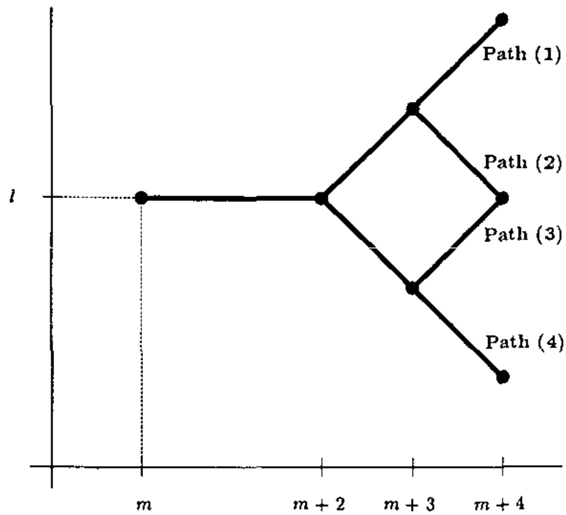


**Figure 1.** The paths for the chain (3.8).

### *3.2. The fundamental braiding matrices*

In this section, it will be proved that the fundamental braiding matrices (those for the field $\phi_1^1$)

$$R_m^{l'l} \equiv \left[ R_{(1,1);(1,1)}^{(l',m+2);(l,m)} \right] \tag{3.6}$$

are in fact independent of $m$. To do so, it is also helpful to consider the braiding matrices

$$\begin{aligned} \chi_{(l,m)} &\equiv \left[ R_{(0,2),(0,2)}^{(l,m+4);(l,m)} \right]_{(l,m+2);(l,m+2)} \\ \gamma_{(l,m)}^{\pm} &\equiv \left[ R_{(1,1),(0,2)}^{(l\pm1,m+3);(l,m)} \right]_{(l\pm1,m+1);(l,m+2)} \\ \theta_{(l,m)}^{\pm} &\equiv \left[ R_{(0,2),(1,1)}^{(l\pm1,m+3);(l,m)} \right]_{(l,m+2);(l\pm1,m+1)} . \end{aligned} \tag{3.7}$$

The logical steps of this proof consists of proving that (i) $R_m^{l\pm2,l}$ is independent of $m$; (ii) $\chi_{(l,m)}$ is independent of both $l$ and $m$; (iii) there exists a normalization of the interpolating fields of $\phi_1^1$ such that $\gamma_{(l,m)}^{\pm}$ and $\theta_{(l,m)}^{\pm}$ are both independent of $l$ and $m$; and (iv) $R_m^{l,l}$ is independent of $m$.

The braid relations (2.4) for the chains of interpolating fields:

$$(l_0,m+4) \overset{(1,1)}{\perp} (l_1,m+3) \overset{(1,1)}{\perp} (l,m+2) \overset{(0,2)}{\perp} (l,m) \tag{3.8}$$

will be used to prove properties (i) and (iv). For each initial sector $(l,m)$, there are four different chains compatible with (3.2), that is four pairs $(l_0,l_1)$: $\{(l+2,l+1),(l,l_1),(l,l-1),(l-2,l-1)\}$ corresponding to the paths labelled (1) to (4) on figure 1. We have summarized in table 1 the relevant indices $\{\beta_1',\beta_1'',\beta_2',\beta_2''\} \sim \{(l_1',m_1'),(l_1'',m_1''),(l_2',m_2'),(l_2'',m_2'')\}$ necessary to specify the braid relations:

**Table 1.** Intermediate sectors for three fields in equation (3.8).

| | Path (1) | Path (2) | Path (3) | Path (4) |
|---|---|---|---|---|
| $(l_1', m_1')$ | $(l+2, m+2)$ | $(l, m+2)$ | $(l, m+2)$ | $(l-2, m+2)$ |
| $(l_1'', m_1'')$ | $(l+1, m+3)$ | $(l\pm 1, m+3)$ | $(l\pm 1, m+3)$ | $(l-1, m+3)$ |
| $(l_2', m_2')$ | $(l+1, m+1)$ | $(l\pm 1, m+1)$ | $(l\pm 1, m+1)$ | $(l-1, m+1)$ |
| $(l_2'', m_2'')$ | $(l+1, m+1)$ | $(l+1, m+1)$ | $(l-1, m+1)$ | $(l-1, m+1)$ |

Property (i) follows immediately from the Yang–Baxter equation (2.4) for paths (1) and (4).

Property (ii) can be obtained similarly; the braid relations for the chains (one path for each $(l, m)$):

$$_{(l,m+6)}\overset{(0,2)}{\perp}{}_{(l,m+4)}\overset{(0,2)}{\perp}{}_{(l,m+2)}\overset{(0,2)}{\perp}{}_{(l,m)}$$

give $\chi_{(l,m)} = \chi_l$, and the two sets of chains:

$$_{(l\pm1,m+5)}\overset{(0,2)}{\perp}{}_{(l\pm1,m+3)}\overset{(0,2)}{\perp}{}_{(l\pm1,m+1)}\overset{(1,1)}{\perp}{}_{(l,m)}$$

give $\chi_{l\pm1} = \chi_l$.

Property (iii) is more delicate. The field $\phi_3^1$ is the only field obtained by fusing $\phi_2^0$ and $\phi_1^1$. The interpolating field $_{(l\pm1,m+3)}\overset{(1,3)}{\perp}{}_{(l,m)}$ can hence be obtained by either

$$_{(l\pm1,m+3)}\underset{x}{\overset{(1,3)}{\perp}}{}_{(l,m)} = {}^R\lim_{\epsilon\to0}\alpha^{\pm}_{(l,m)}\,{}_{(l\pm1,m+3)}\underset{x+\epsilon}{\overset{(0,2)}{\perp}}{}_{(l\pm1,m+1)}\underset{x}{\overset{(1,1)}{\perp}}{}_{(l,m)} \tag{3.9a}$$

or

$$_{(l\pm1,m+3)}\underset{x}{\overset{(1,3)}{\perp}}{}_{(l,m)} = {}^R\lim_{\epsilon\to0}\beta^{\pm}_{(l,m)}\,{}_{(l\pm1,m+3)}\underset{x+\epsilon}{\overset{(1,1)}{\perp}}{}_{(l,m+2)}\underset{x}{\overset{(0,2)}{\perp}}{}_{(l,m)}. \tag{3.9b}$$

where ${}^R\lim$ regularizes the divergent Wilson product [3]. The $\alpha^{\pm}_{(l,m)}$ and $\beta^{\pm}_{(l,m)}$ are constants that depend on the normalization of the interpolating fields. Since the exchange matrices depend only on the relative position of the arguments of the braided fields, we can use either of these two expressions to express $R^{(l\pm1,m+5);(l,m)}_{(1,3);(0,2)}$ and $R^{(l\pm2,m+4);(l,m)}_{(1,1);(1,3)}$. Doing so, we conclude that the $\gamma^{\pm}_{(l,m)}$ are independent of $l$ and $m$ if the ratios are

$$\frac{\beta^{+}_{(l,m)}}{\alpha^{+}_{(l,m)}} \qquad \text{and} \qquad \frac{\beta^{-}_{(l,m)}}{\alpha^{-}_{(l,m)}}$$

are also independent of $l$ and $m$. Similarly, calculating $R^{(l\pm1,m+5);(l,m)}_{(0,2);(1,3)}$ and $R^{(l\pm2,m+4);(l,m)}_{(1,3);(1,1)}$, property (iii) for $\theta^{\pm}_{(l,m)}$ follows, again under the same assumption on the earlier ratios. The condition on these ratios can be achieved by recursively normalizing the interpolating fields.

Using properties (i)–(iii), the last property now appears to be an immediate consequence of the braiding equation (2.4) for the paths (2) and (3) of the chain (3.8) (see table 1). This ends the proof.

Since the fundamental exchange matrices are independent of $m$, we only need the $l$ part of the fusion rules (3.4) when writing relation (2.4) for $\alpha_1 = \alpha_2 = \alpha_3 = (1,1)$. Hence $R^{l',l}$ verifies the same Yang–Baxter equations as those verified by the matrices $[R^{l'+1,l+1}]$ of [4]. There the general solution was found to be

$$R^{0,0} = R^{NN} = \eta$$

$$R^{l-1,l+1} = R^{l+1,l-1} = \eta\omega$$

$$[R^{ll}]_{l\mp1,l\mp1} = \eta(-\omega)^{1/2}\begin{pmatrix} -(-\omega)^{(l+1)/2}\dfrac{s(1)}{s(l+1)} & \lambda_l^{-1}\sqrt{\dfrac{s(l)s(l+2)}{s^2(l+1)}} \\ \lambda_l\sqrt{\dfrac{s(l)s(l+2)}{s^2(l+1)}} & (-\omega)^{-(l+1)/2}\dfrac{s(1)}{s(l+1)} \end{pmatrix} \tag{3.10}$$

with

$$(-\omega) = \exp\frac{2\pi i p}{N+2} \qquad \text{with } p \text{ and } N+2 \text{ coprime}$$

$$s(l) = \sin\left(\frac{l\pi p}{N+2}\right)$$

where $\lambda_l$ and $\eta$ are complex numbers left undetermined by equation (2.4). (In [4], the identity family was labelled by (1) instead of (0) as here, and the number of families was $q-1$ instead of $N+1$.) This is the solution of the braid equations for the field $\phi_1^1$.

### 3.3. Solution of the phase condition

As the final step in constructing the exchange algebra, we solve phase condition (2.3) for the scheme (3.4). Knowing $R^{l'l}$, these conditions provide us with constraints on the spectrum of conformal weights of the families $[\phi_m^l]$.

For the case at hand, the braiding matrices $R^{l'l}_{(m)}$ do not depend on $m$ as shown earlier and the phase condition is

$$\sum_{l_1'}[R^{(l_0,l)}]_{l_1,l_1'}\ [R^{(l_0,l)}]_{l_1',l_1''}\exp\left(2\pi i\left[h^{l_1}_{m+1} + h^{l_1'}_{m+1} - h^{l_0}_{m+2} - h^l_m\right]\right) = \delta_{l_1,l_1''}.$$

For each $l$ and $m$ with the same parity, this equation should be written for $l_0 = l \pm 2$ (and then $l_1 = l_1' = l_1'' = l \pm 1$) and for $l_0 = l$ (and then $l_1 = l_1'' = l \pm 1$ and the sum over $l_1'$ assumes both values $l \pm 1$). The cases $l = 0$ or $l = N$ have to be treated separately. The resulting equations obtained using (3.10) are (the notation $e(h) = \exp(2\pi i h)$ is used):

$$l_0 = l = 0: \qquad e(h^0_m + h^0_{m+2}) = \eta^2 e(2h^1_{m+1}) \tag{3.11}$$

$$l_0 = l = N: \qquad e(h^N_m + h^N_{m+2}) = \eta^2 e(2h^{N-1}_{m+1}) \tag{3.12}$$

$l_0 = l \pm 2$ (with $l_0 \neq 2$ and $l_0 \neq N+2$) :

$$e\left(h^l_m + h^{l\pm2}_{m+2}\right) = (\eta\omega)^2 e(2h^{l\pm1}_{m+1}) \tag{3.13}$$

$l_0 = l = 1, \ldots, N-1$ :

$$l_1 = l_1'' = l+1: \qquad e(h^l_m + h^l_{m+2}) = \eta^2(-\omega)e\left(h^{l+1}_{m+1} + h^{l-1}_{m+1}\right) \tag{3.14}$$

$$l_1 = l+1 \text{ and } l_1'' = l-1: \quad e(h^{l+1}_{m+1}) = (-\omega)^{l+1} e(h^{l-1}_{m+1}). \tag{3.15}$$

The cases ($l_1 = l_1'' = l-1$) and ($l_1 = l-1$, $l_1'' = l+1$) give equations equivalent to (3.14) and (3.15).

We now proceed to the solution of equations (3.11)–(3.15). Equation (3.15) allows us to write $e(h_m^l)$ in terms of $e(h_m^0)$ of $e(h_m^1)$ depending on the parity of $l$. The result is

$$e(h_m^l) = \begin{cases} (-\omega)^{\frac{l}{4}(l+2)} e(h_m^0) & \text{for } l \text{ even} \\ (-\omega)^{\frac{l}{4}(l+2)-\frac{3}{4}} e(h_m^1) & \text{for } l \text{ odd.} \end{cases} \tag{3.16}$$

Moreover equation (3.16) can now be used to express equations (3.12)–(3.14) only in terms of $e(h_m^0)$ and $e(h_m^1)$. In fact, a direct check shows that equations (3.13) and (3.14) are all equivalent to either one of the following equations:

$$\begin{aligned} e(h_m^0 + h_{m+2}^0) &= \eta^2 e(2h_{m+1}^1) \\ e(h_m^1 + h_{m+2}^1) &= \eta^2 (-\omega)^3 e(2h_{m+1}^0). \end{aligned} \tag{3.17}$$

(Notice that the first one is precisely (3.11).) Equation (3.12) becomes

$$(-\omega)^{N+2} e(h_m^0 + h_{m+2}^0) = \eta^2 e(2h_{m+1}^1) \qquad \text{if } N \text{ is even}$$

$$(-\omega)^{N-1} e(h_m^1 + h_{m+2}^1) = \eta^2 e(2h_{m+1}^0) \qquad \text{if } N \text{ is odd.}$$

Comparing with equations (3.17), we are forced to set

$$(-\omega)^{N+2} = 1$$

or, in other words,

$$(-\omega) = \mathrm{e}^{2\pi \mathrm{i} p/(N+2)} \qquad p \in \mathbb{Z}. \tag{3.18}$$

Hence, the set of equations (3.11)–(3.15) is equivalent to equations (3.16)–(3.18).

By multiplying the first equation of (3.17) for two consecutive values of $m$ and $(m+2)$, we get

$$e(h_{m+4}^0) = \eta^8 (-\omega)^6 e(2h_{m+2}^0 - h_m^0). \tag{3.19}$$

Similarly, we can obtain the following equation for the odd values of $m$:

$$e(h_{m+3}^1) = \eta^8 (-\omega)^6 e(2h_{m+1}^1 - h_{m-1}^1). \tag{3.20}$$

To start the recursive solution of these equations, we need to fix the first values of $h_m^l$s. Since the family $[\phi_0^0]$ contains the identity, $e(h_0^0) = 1$. Let us write $h_1^1 = \theta$. Then, by equation (3.17), we get: $e(h_2^0) = \eta^2 e(2\theta)$ and $e(h_3^1) = \eta^6 (-\omega)^3 e(3\theta)$. These values are sufficient to obtain the full solution of equations (3.19) and (3.20):

$$e(h_m^0) = \eta^{m(m-1)} (-\omega)^{\frac{3}{4}m(m-2)} e(m\theta) \tag{3.21}$$

and

$$e(h_{m+1}^1) = \eta^{m(m-1)} (-\omega)^{\frac{3}{4}m(m-2)+\frac{3}{4}} e(m\theta).$$

Further constraints on $\eta$ and $\theta$ can be obtained by using the fact that the families $[\phi^l_m]$ and $[\phi^l_{m+2N}]$ have the same spectrum modulo $\mathbb{Z}$. These cyclic conditions $e(h^0_m) = e(h^0_{m+2N})$ and $e(h^1_{m+1}) = e(h^1_{m+1+2N})$ lead to the following equation:

$$\left[\eta^4(-\omega)^3\right]^{N^2+mN-N} \eta^{2N} e(2N\theta) = 1$$

which should hold for any $m$. Hence $[\eta^4(-\omega)^3]^N = 1$ and

$$\eta^4(-\omega)^3 = e(\kappa/N) \tag{3.22}$$

for a certain $\kappa \in \mathbb{Z}$. Hence, $\eta^2$ can be expressed as

$$\eta^2 = e\left(\frac{\kappa}{2N} - \frac{3p}{2(N+2)} + \epsilon\right) \tag{3.23}$$

where $\epsilon = 0$ or $\frac{1}{2}$. This leaves the constraint $(\eta^2 e(2\theta))^N = 1$ which can be solved for $\theta$:

$$h^1_1 = \theta = \frac{(2\tau - \kappa - 2\epsilon N)(N+2) + 3pN}{4N(N+2)} \tag{3.24}$$

where a new parameter $\tau \in \mathbb{Z}$ has been introduced.

Putting (3.16), (3.21) and (3.24) together, the spectrum becomes

$$e(h^l_m) = e\left(\frac{pl(l+2)}{4(N+2)} + \frac{\tau m}{2N} + \left(\frac{\kappa}{4N} + \frac{\epsilon}{2}\right) m(m-2)\right).$$

There is some redundancy in the choice of the parameters $\tau, \kappa$ and $\epsilon$. Since $\kappa \in \mathbb{Z}$ and $\epsilon = 0$ or $\frac{1}{2}$, the new parameter $\hat{\kappa}$ given by

$$\frac{\hat{\kappa}}{4N} = \left(\frac{\kappa}{4N} + \frac{\epsilon}{2}\right)$$

turns out to always be integer. Moreover, defining $\hat{\tau} = \tau - \hat{\kappa}$, the full solution of the phase condition (2.3) depends on the parameters, $p, \hat{\tau}$ and $\hat{\kappa} \in \mathbb{Z}$ as

$$e(h^l_m) = e\left(\frac{pl(l+2)}{4(N+2)} + \frac{\hat{\tau} m}{2N} + \frac{\hat{\kappa} m^2}{4N}\right). \tag{3.25}$$

All the possible spectra of the theories having a field $[\phi^1_1]$ with fusion rules as in (3.4) are thus determined by the structure of the exchange algebra (up to integers).

## 4. Concluding remarks

Parafermionic theories were introduced by Zamolodchikov and Fateev [6] as candidate conformal theories for reproducing the critical exponents of the $\mathbb{Z}_N \times \hat{\mathbb{Z}}_N$ critical autodual systems on a two-dimensional lattice. For our purpose the field content of these theories is summarized by the following enumeration of conformal families:

$$\{[\phi^{l\bar{l}}_{m\bar{m}}];\ l, \bar{l} = 0, \ldots, N,\ \forall m, \bar{m} \text{ such that } m = l \bmod 2, \bar{m} = \bar{l} \bmod 2\}. \tag{4.1}$$

Besides determining the transformation properties of a field under the action of $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$ its charges $m$ and $\bar{m}$ also specify the monodromy properties of its correlation functions. If $\phi_1(z_1, \bar{z}_1)$ has charges $(m_1, \bar{m}_1)$ and $\phi_2(z_2, \bar{z}_2)$ has charges $(n, \bar{n})$, then any correlation function containing the product $\phi_1\phi_2$ picks up a phase factor $e^{-\pi i(mn-\bar{m}\bar{n})/N}$ under the following manipulation: $z_1$ circles $z_2$ clockwise while $\bar{z}_1$ circles $\bar{z}_2$ anticlockwise. (Note that as usual $z$ and $\bar{z}$ are independent complex variables, $z = x + iy$ and $\bar{z} = x - iy$, $x, y \in \mathbb{C}$, with the Minkowskian section corresponding to $x = x^0 \in \mathbb{R}$ and $y = -ix^1, x^1 \in \mathbb{R}$.) Two fields are said to be mutually local when this phase is trivial.

If $(h^l_m, \bar{h}^{\bar{l}}_{\bar{m}})$ are the dimensions of the primary field $\phi^{l\bar{l}}_{m\bar{m}}$ then we have $h^l_{m+2N} = h^l_m \bmod \mathbb{Z}$, $\bar{h}^{\bar{l}}_{\bar{m}+2N} = \bar{h}^{\bar{l}}_{\bar{m}} \bmod \mathbb{Z}$ and

$$h^l_m = \begin{cases} \dfrac{l(N-l)}{2N(N+2)} + \dfrac{(l-m)(l+m)}{4N} & \text{for } -l \leqslant m \leqslant l \\[2ex] \dfrac{l(N-l)}{2N(N+2)} + \dfrac{(m-l)(2N-l-m)}{4N} & \text{for } l \leqslant m \leqslant 2N-l. \end{cases} \tag{4.3}$$

Similar expressions hold for $\bar{h}^{\bar{l}}_{\bar{m}}$.

The fusion rules for the parafermionic theories were given by Gepner and Qiu [9]:

$$[\phi^{l_1\bar{l}_1}_{m_1\bar{m}_1}][\phi^{l_2\bar{l}_2}_{m_2\bar{m}_2}] = \sum_{l=|l_1-l_2|}^{\min\{l_1+l_2, 2N-(l_1+l_2)\}} \ \sum_{\bar{l}=|\bar{l}_1-\bar{l}_2|}^{\min\{\bar{l}_1+\bar{l}_2, 2N-(\bar{l}_1+\bar{l}_2)\}} [\phi^{l\bar{l}}_{m_1+m_2, \bar{m}_1+\bar{m}_2}] \tag{4.4}$$

where the sums run over $l$ and $\bar{l}$ such that $l = l_1 - l_2 \bmod 2$ and $\bar{l} = \bar{l}_1 - \bar{l}_2 \bmod 2$. Certain conformal families on the right-hand side must have a vanishing structure constant in order for (4.4) to behave properly under rotations. For example, specializing to the case where $l_i = \bar{l}_i$, $m_i = \bar{m}_i$, $i = 1, 2$, we have two mutually local fields on the left. The spin of the fields on the right-hand side must then vanish ($h^l_m = h^{\bar{l}}_m$) which implies $l = \bar{l}$ for a general $N$.

Now identify $[\phi^l_m]$ of the previous section with $[\phi^{ll}_{mm}]$ for the proper domain of $m$. The fusion rules (4.4) are then seen to be identical to (3.2) provided this comment is taken into account. Moreover, if we set $p = 1, \hat{r} = -1$ and $\hat{\kappa} = 0$, the spectrum (3.25) matches the earlier one.

A comment should be made on the current algebra acting on the parafermionic theories and its relationship with the chiral algebra discussed in section 2. Parafermionic theories contain conserved parafermionic currents. These generate an algebra (intimately related the **Z**-algebra of Lepowsky and Wilson [10]) which is not a Lie algebra. The representations of this algebra contain weights which are not necessarily spaced by integers (see [6] and [9]–[10]). Hence, identifying this parafermionic algebra with a chiral algebra would not have been suitable when discussing the braiding properties of a parafermionic family (associated with an irreducible module) since fields whose weight differences are not integer have different monodromy properties.

One could remark that, even though we have only used local fields $\phi^l_m = \phi^{ll}_{mm}$ throughout the computations, the spectrum also contains the conformal weights of the (non-local) parafermionic currents. This might seem *a priori* a miracle. However it simply stems from the fact that the light-cone interpolating fields of $[\phi^1_1]$ are also the building blocks of these currents; only that the various coefficients $g^{(\alpha,\beta),(\bar{\alpha},\bar{\beta})}_{(h_\Phi, \bar{h}_\Phi)}$ are different for the latter (see section 2).

Finally (and this is probably the most striking result), it should be emphasized that no explicit knowledge of either the Virasoro or parafermionic algebras was used in section 3. The only hypothesis characterizing the theories were:

(i) the existence of a field $[\phi_1^1]$ with the specific fusion rules (3.4); and

(ii) the decomposition of the Hilbert space as

$$\mathcal{H} = \bigoplus_{\substack{0\leqslant l\leqslant N \\ 0\leqslant m\leqslant 2N-1 \\ m=l \bmod 2}} \mathcal{H}_{(l,m)}.$$

It indicates that the classification of conformal quantum field theories using, for example, the rational equations from Moore and Seiberg [1] or the quantum groups language might follow a lattice-theoretic approach where the (periodic) lattice would represent the decomposition of the Hilbert space and where the basic periods would encompass the fusion rules of the interpolating fields of a well-chosen subset of fields.

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